Math 8 Homework 5

1 Mathematical Induction and the Well Ordering Principle

(a) Prove that for any integer $n \geq 1$ we have
$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

(b) Prove that for any integer $n \geq 1$ we have
$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$$

(c) Prove that 3 divides $4^n - 1$ for all $n \geq 1$.

(d) Prove that for all $n \geq 1$,
$$\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots < \frac{1}{n!}.$$
Hint: For the base case use facts about e.

(e) Suppose x is a complex number so that $x + 1/x \in \mathbb{Q}$. Prove that $x^n + 1/x^n \in \mathbb{Q}$ for each $n \in \mathbb{N}$.

(f) Suppose $f : \mathbb{Q} \to \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ for all rationals x, y. Prove there is a $c \in \mathbb{R}$ so that $f(x) = cx$ for all rational x. Hint: Do this in phases. First prove it for integers x. Then try to handle reciprocals of integers. Finally turn to general rational numbers.

(g) Let $f : \mathbb{Z} \to \mathbb{N}$ have the following property: for any $n \in \mathbb{Z}$ we have that $f(n)$ is the average of $f(n-1)$ and $f(n+1)$. Prove f is constant. Hint: The image of f is a subset of $\mathbb{N}$. Use the well ordering principle.

2 The Pigeonhole Principle

Note: Most of these are very hard until you see how to proceed. Don't be discouraged; just relax and think.

(a) In a room of 100 people, some pairs shake hands and some don't. No two people shake hands more than once and nobody shakes hands with herself. Prove some two people shook the same number of hands.

(b) 5 points are chosen within an equilateral triangle of side length 1. Prove some two are less than 1/2 apart.

(c) Let $x_1, x_2, \ldots, x_7$ be arbitrary real numbers. Prove there are two among them, say x_i and x_j, so that
$$0 \leq \frac{x_i - x_j}{1 + x_i x_j} \leq \frac{1}{\sqrt{3}}.$$
Hint: The tangent function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is a bijection. Find a relevant trigonometric identity.

(d) In a perfect world, any pair of people are either close friends or total strangers. Ramsey finds himself in a room with 5 other people. Prove that there are either 3 mutual friends or 3 mutual strangers among them.

(e) In a less-than-perfect world, any 2 people are either close friends, total strangers, or vicious enemies. Ramsey finds himself in a room with 16 other people. Prove that there are either 3 mutual friends, strangers, or enemies. Hint: use the previous parts in a clever way!

(f) In a super chaotic world, any two people have relationship–type 1, 2, 3,…, or n. Let's define R_n to be the least number of people it takes before we are guaranteed a trio of the same relationship–type. Prove $R_2 = 6$.

(g) Prove that for $n \geq 2$
$$R_{n+1} \leq (n+1)(R_n - 1) + 2.$$
Hint: Repeat the same argument from the previous parts.

(h) Use the previous results and induction to prove for $n \geq 2$
$$R_n \leq \lfloor n!e \rfloor + 1.$$
Hint: Use one of the results from the top of the page to help with the induction.